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ABSTRACT

This paper addresses the issue of computing the asymptotic worst-case of lower bounds for the Bin Packing Problem. We introduce a general result that allows to bound the asymptotic worst-case performance of any lower bound for the problem and to derive for the first time the asymptotic worst-case of the well-known bound L_3 by Martello and Toth. We also show that the general result allows to easily derive the asymptotic worst-case of several lower bounds proposed in the literature.

Keywords: Bin Packing, lower bounds, asymptotic worst-case analysis.

RÉSUMÉ

Le sujet de cet article est la problématique du calcul de la borne asymptotique en pire cas pour le problème de « Bin Packing ». Nous présentons un résultat général qui permet de dériver une limite pour la performance asymptotique en pire cas pour n'importe quelle borne inférieure pour ce problème. Ce résultat nous permet de présenter pour la première fois une telle analyse pour la très connue borne L_3 proposée par Martello et Toth, ainsi que pour plusieurs autres bornes inférieures proposées dans la littérature.

Mots clés : « Bin Packing », bornes inférieures, analyse asymptotique en pire cas.

1 Introduction

Given a set I of n items, each with a volume v_i, $i \in I$, and containers of fixed capacity V called bins, the *Bin Packing* (BP) problem consists in assigning each item to a bin, such that the sum of the volumes of the items within the same bin is not greater than the capacity of the bin, and the number of bins is minimal. According to the typology introduced by Wäscher et al., the problem can be characterized as the *One-Dimensional Single Bin-Size Bin Packing Problem* [10]. The BP problem is known to be NP-hard [7].

The contribution of this paper is a general result on the analysis of the asymptotic worst-case of the BP lower bounds. The theorem we provide allows to derive a lower bound of the asymptotic worst-case performance of any lower bound of the BP problem. By means of this theorem, we provide for the first time the asymptotic worst-case ratio of the L_3 bound by Martello and Toth [7]. Moreover, the asymptotic worst-case ratio of the bounds L_{2LLM} by Labbé, Laporte, and Mercure [6], L_2 by Martello and Toth [7], and $L_{FS}^{\star(p)}$ by Fekete and Schepers [4, 9] can be easily derived as well.

The paper is organized as follows. In Section 2, we recall existing lower bounds for the BP problem. Section 3 introduces the new results, proves the asymptotic worst-case of L_3, and generalizes the derivation of the asymptotic worst-case of L_2, L_{2LLM}, and $L_{FS}^{\star(p)}$.

2 Lower Bounds for the BP Problem

Let us assume, without loss of generality, that the volume of item i, v_i, $i \in 1, ..., n$, is at most equal to the bin capacity V. One may then normalize the volumes onto the interval $[0, 1]$, where 1 corresponds to the bin capacity V. Define the following set of variables:

x_{ij} binary variable that is equal to 1 if item i is assigned to bin j, 0 otherwise;

y_j binary variable that is equal to 1 if bin j contains at least one item, 0 otherwise.

The BP problem can then be formulated as follows:

$$min \quad \sum_{j=1}^{n} y_j \tag{1}$$

$$\sum_{i=1}^{n} v_i x_{ij} \leq y_j \quad j = \{1, ..., n\} \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = \{1, ..., n\} \tag{3}$$

$$y_j \in \{0, 1\}, \forall j \tag{4}$$

$$x_{ij} \in \{0, 1\}, \forall i, \forall j. \tag{5}$$

The objective function (1) minimizes the number of used bins. Constraints (2) enforce the bin capacity restrictions and force variables y_j to

assume value 1 if at least one item i is assigned to bin j. Equations (3) force to load each item into just one bin. In the following, we review the existing lower bounds on the optimum value of (1) and discuss their performance.

Elementary bounds can be derived from the continuous, Lagrangian, and surrogate relaxations of the model (1) - (5).

It can be proved [7] that bounds obtained from the continuous and surrogate relaxations are dominated by the lower bound computed as follows:

$$L_1(I) = \left\lceil \sum_{i=1}^{n} v_i \right\rceil . \tag{6}$$

The bound L_1 can be computed in $O(n)$ time and it has a worst case performance of $\frac{1}{2}$ ([5, 7]). The bound obtained from the Lagrangian relaxation of constraints (2) is also dominated by L_1 [7].

A better bound was introduced by Martello and Toth by focusing on the items with $v_i > \frac{1}{2}$ [7, 8]. The main idea is that, given a value $\varepsilon \in \left[0, \frac{1}{2}\right]$, one assumes that the items with a volume $v_i < \varepsilon$ are used to completely fill the bins in which the items with volume $v_i > 1 - \varepsilon$ are loaded. Let us define $I(a, b) = \{i \in I | a < v_i \leq b\}$. The bound can be formulated as

$$\begin{aligned} L_2(I) &= \max_{\epsilon \in \left[0, \frac{1}{2}\right]} \left(|I(1-\epsilon, 1)| + L_1\left(I_2(\epsilon)\right) \right) \\ I_2(\epsilon) &= \left\{ i \in I \mid \epsilon \leq v_i \leq 1 - \epsilon \right\} . \end{aligned} \tag{7}$$

This bound has an asymptotic worst-case ratio $r(L_2) = \frac{2}{3}$. It can be computed in time $O(n \lg n)$ if the items are unsorted and in $O(n)$ if the items are sorted by non-increasing volumes [7].

Labbé, Laporte, and Mercure improved L_2 by considering the set of items in $I(\frac{1}{3}, \frac{1}{2})$ that cannot be loaded together with the items in $I(\frac{1}{2}, 1)$ [6]. Assume items are sorted by non-decreasing volumes. Labbé, Laporte, and Mercure then attempt to assign the items in $I(\frac{1}{3}, \frac{1}{2})$ according to an adapted First Fit procedure [2]. The procedure first assigns each item in $I(\frac{1}{2}, 1)$ to a different bin. Items in $I(\frac{1}{3}, \frac{1}{2})$ are then assigned to the same bins, in non-decreasing volume order, by pairing them to items in $I(\frac{1}{2}, 1)$ such that the residual capacity of the bin is minimized.

Let I_3 the set of the items with volume $1/3 < v_i \leq 1/2$ which cannot be assigned to a bin according to this procedure, i.e., they require at least $\lceil |I_3|/2 \rceil$ additional bins. More formally, this bound, called L_{2LLM}, can be expressed as follows:

$$\begin{aligned} L_{2LLM}(I) = \left\lceil \frac{|I_3|}{2} \right\rceil + \max_{\epsilon \in \left[0, \frac{1}{2}\right]} \left(|I(1-\epsilon, 1)| + \right. \\ \left. \max\left\{0, L_1\left(I_2(\epsilon)\right) - \lceil |I_3|/2 \rceil\right\}\right) \\ I_2(\epsilon) = \left\{ i \in I \mid \epsilon \leq v_i \leq 1 - \epsilon \right\} . \end{aligned} \tag{8}$$

Recently, Bourjolly and Rebetez proved that this bound has an asymptotic worst-case ratio $r(L_{2LLM}) = \frac{3}{4}$ [1]. L_{2LLM} can be computed in time

$O(n \lg n)$ if the items are unsorted and in $O(n)$ if the items are sorted by non-increasing volumes.

A new class of lower bounds has been recently introduced by Fekete and Schepers [4]. They base their bound on the use of item volumes modified by Dual Feasible Functions, where a function $u : [0,1] \longrightarrow [0,1]$ is called dual feasible if, given a finite set F such that $\sum_{x \in F} x \leq 1$, $\sum_{x \in F} u(x) \leq 1$.

The bound $L_{FS}^{*(p)}$ proposed by Fekete and Schepers is defined as follows:

$$L_{FS}^{*(p)} = \max \left\{ L_2(I), max_{k=2,\ldots,p} L_2^{(k)}(I) \right\},$$

where

$$L_2^{(k)}(I) = \max_{\epsilon \in \left[0, \frac{1}{2}\right]} L_1\left(u^{(k)} \circ U^{(\epsilon)}(I)\right)$$

$$u^{(k)}(x) : [0,1] \rightarrow [0,1], u^{(k)}(x) = \begin{cases} x & x(k+1) \in \mathbb{N} \\ \frac{\lfloor x(k+1) \rfloor}{k} & otherwise \end{cases}$$

$$U^{(\epsilon)}(x) : [0,1] \rightarrow [0,1], U^{(\epsilon)}(x) = \begin{cases} 1 & x > 1 - \epsilon \\ x & \epsilon \leq x \leq 1 - \epsilon \\ 0 & x < \epsilon \end{cases}$$

$$f(x) \circ g(x) = g(f(x)).$$

Fekete and Schepers proved that $L_{FS}^{*(p)}$ has an asymptotic worst-case performance of $\frac{3}{4}$ and can be computed in $O(n)$ if the items are sorted by non-increasing volumes.

One of the tightest bounds for the BP problem, known as L_3, has been developed by Martello and Toth [7]. The authors use a reduction procedure called $MTRP$ to build the non-dominated bins in an optimal solution that contain at most 3 items (if such bins exist). The corresponding items are then removed and the L_2 bound is applied to the remaining ones.

Define a *Feasible Set* $F \in FS$ as a set of items for which the sum of volumes is not greater than the capacity of a bin. Then, if we consider an item i of an instance I of the BP problem where items are sorted by non-increasing volumes, the maximum cardinality of the feasible sets $F \in FS$ that include i in an optimal solution is

$$F_{i,\max} = 1 + \begin{cases} 0 & \text{if } v_i + v_N > 1 \\ \max_{k \in I \setminus \{i\}} \left\{ k : \sum_{j=1}^{k} v_{N-j+1} \leq 1 - v_i \right\} & \text{otherwise.} \end{cases}$$

Consider an instance I of the BP problem with items sorted by volumes in non-increasing order. The $MTRP$ procedure is then based on the following dominance criteria:

1. If $F_{1,\max} = 1$, then there exists an optimal solution containing $F = \{v_1\}$, $F \in FS$.

2. If $F_{1,\max} = 2$, then there exists an optimal solution containing $F = \{1, j\}$, $F \in FS$, where

$$j = \arg \min_{j : \{1,j\} \text{ is a } FS} \{1 - (v_1 + v_j)\}.$$

3. If $F_{1,\max} = 3$, then there exists an optimal solution containing the feasible set:

   (a) $F = \{1, k\}$, $k = \arg\min_{\{1,k\} \in FS} \{1 - (v_1 + v_k)\}$ if $\nexists \{i, j\} \in I$: $1 - (v_1 + v_i + v_j) \leq 1 - (v_1 + v_k)$, $\{1, i, j\} \in FS$;

   (b) $F = \{1, i, j\}, \{i, j\} = \arg\min_{\{1,i,j\} \in FS, i<j} \{1 - (v_1 + v_i + v_j)\}$, if $i = \arg\min_{\{1,i\} \in FS} \{1 - (v_1 + v_i)\}$ and $j - i \leq 2$;

   (c) $F = \{1, i, j\}, \{i, j\} = \arg\min_{\{1,i,j\} \in FS, i<j} \{1 - (v_1 + v_i + v_j)\}$, if $i = \arg\min_{\{1,i\} \in FS} \{1 - (v_1 + v_i)\}$ and $v_1 + v_{j-1} + v_{j-2} > 1$.

L_3 is defined by applying the procedure $MTRP$ for $n-1$ iterations. At the beginning, $MTRP$ is applied to the instance I. A new instance is obtained at each iteration through the removal of the item with the smallest volume. The value of the lower bound is the maximum of the values obtained by the $n-1$ iterations. The $MTRP$ can be applied in $O(n^2)$ and thus L_3 is $O(n^3)$. The bound L_3 is quite time consuming. Computational experiments show, however, that it is very tight [7]. No asymptotic worst-case result is known for the bound L_3.

3 Worst-Case Analysis of BP Lower Bounds

Let I be an instance of the BP problem and $L(I)$ a lower bound on the optimum of I. The performance of L for the instance I is defined as $\frac{L(I)}{OPT(I)}$, where $OPT(I)$ is the optimum. A theoretical measure of the performance of L is obtained through the worst-case analysis.

Definition 1 (Asymptotic worst-case) *Let I be a generic BP instance, L a lower bound for the BP problem, and $OPT(I)$ the optimum of I. Given a positive integer s, the* asymptotic worst-case *of L is given by*

$$r_\infty(L) = lim_{s \to \infty} sup \left\{ \frac{L(I)}{OPT(I)} \mid \forall I \text{ with } OPT(I) \geq s \right\}.$$

The asymptotic worst-case analysis is generally difficult to perform. In the following, we prove a general result that reduces the complexity of computing the asymptotic worst-case of BP lower bounds. We then use it to derive the asymptotic worst case of the L_3 bound.

Theorem 2 *Given a value $k > 1, k \in \mathbb{N}$ and a lower bound LB of the BP problem such that*

$$LB(I') = OPT(I'), \forall I' : \min_{i \in I'}(v_i) > \frac{1}{k},$$
$$LB(I) \geq L_1(I), \ \forall I,$$
$$LB(I^*) \leq LB(I), \forall I,$$

where L_1 is the bound given by (6) and I^* is the sub-instance of I defined as $I^* = \{i \in I : v_i > \frac{1}{k}\}$, the asymptotic worst-case of LB is

$$r_\infty(LB) \geq \frac{k}{k+1}.$$

Proof. We know by hypothesis that the bound LB solves to optimality the instances of the BP problem in which all the items have a volume larger than $\frac{1}{k}$.

Consider an instance I. It can be split into two sub-instances I_1 and I_2 such that

$$I_1 = \left\{ i \in I : v_i > \frac{1}{k} \right\}$$
$$I_2 = \left\{ i \in I : v_i \leq \frac{1}{k} \right\}.$$

We then need exactly

$$m = LB(I_1) = OPT(I_1)$$

bins to load all the items of the instance I_1. Index these bins by the integers 1 through m. Recall that the First Fit Decreasing heuristic builds a solution by considering the items sorted by non-increasing volumes and putting them into the first partially loaded bin with sufficient residual capacity or, if the item does not fit into the existing bins or no bin has been used yet, into a new bin [7]. Using this heuristic, either the items $i \in I_2$ are assigned to the existing bins or new bins are added, obtaining a total amount of bins $s \geq m$.

Case 1: $s \leq \frac{k+1}{k}m$. If $s \leq \frac{k+1}{k}m$, then

$$OPT(I) \leq s \leq \frac{k+1}{k}m \leq \frac{k+1}{k}LB(I)$$
$$LB(I) \geq \frac{k}{k+1}OPT(I) > \frac{k}{k+1}OPT(I) - 1. \tag{9}$$

Case 2: $s > \frac{k+1}{k}m$. If $s > \frac{k+1}{k}m$, then $\frac{k}{k+1}s > m$. Let α be the largest residual capacity of the first m bins, i.e.

$$\alpha = \max_{j \in [1,m]} \left\{ 1 - \sum_{i \in F_j} v_i \right\},$$

where F_j is the feasible set of the $j-th$ bin. The items to be put into the bins $m+1$ to s have volumes $\alpha < v_i \leq \frac{1}{k}$. Then, at least one of these items is in bin s and each of the bins $j \in \{m+1, ..., s-1\}$ contains at least k items.

With respect to the bound LB, we know that

$$LB(I) \geq L_1(I) = \lceil (1-\alpha)m + k\alpha(s-1-m) + \alpha \rceil \tag{10}$$
$$= \lceil [1-(k+1)\alpha]m + k\alpha(s-1) + \alpha \rceil. \tag{11}$$

Consider the two cases:

Sub-case 1: $\alpha > \frac{1}{k+1}$. The expression (11) becomes

$$\begin{aligned} LB(I) &\geq \lceil [1-(k+1)\alpha] m + k\alpha(s-1) + \alpha \rceil \\ &> \left\lceil [1-(k+1)\alpha] \frac{ks}{k+1} + k\alpha(s-1) + \alpha \right\rceil \\ &= \left\lceil \frac{k}{k+1} s + (1-k)\alpha \right\rceil \geq \frac{k}{k+1} OPT(I) - 1. \end{aligned} \tag{12}$$

Sub-case 2: $\alpha \leq \frac{1}{k+1}$. Consider the residual capacity in the bins $j \in \{m+1, ..., s-1\}$

$$\beta_j = 1 - \sum_{i \in F_j} v_i.$$

Given $v_{i \in F_j} \leq 1/k, j \in \{m+1, \ldots, s-1\}$, $\beta_j \leq \frac{1}{k}$.
If $\beta_j \leq \frac{1}{k+1}$, $\forall j \in \{m+1, \ldots, s-1\}$, the following inequalities hold

$$LB(I) \geq L_1(I) \geq \frac{k}{k+1}(s-1) \geq \frac{k}{k+1} OPT(I) - 1.$$

Otherwise, let us consider j^* as the first bin such that $\beta_{j^*} > \frac{1}{k+1}$. By the First Fit Decreasing heuristic, all the items i loaded into the bins $j \in \{j^*, ..., s-1\}$ must have their volumes $v_i > \beta_{j^*} > \frac{1}{k+1}$. Since each of these bins contains at least k items,

$$\beta_j < \frac{1}{k+1}, j \in \{j^* + 1, ..., s-1\}.$$

Then, j^* is the only bin with $\beta_j > \frac{1}{k+1}$. If we consider the bin s, there must be at least one item with $v_i > \beta_{j^*} > \frac{1}{k+1}$. Consequently

$$\begin{aligned} LB(I) \geq L_1(I) &\geq \frac{k}{k+1}(s-2) + \frac{k-1}{k} + \frac{1}{k+1} \\ &> \frac{k}{k+1} OPT(I) + \left(\frac{1-2k}{k+1} + \frac{k-1}{k} \right) \\ &> \frac{k}{k+1} OPT(I) + \left(\frac{1-2k}{k+1} + \frac{k-1}{k+1} \right) \\ &= \frac{k}{k+1} OPT(I) - \frac{k}{k+1} \\ &\geq \frac{k}{k+1} OPT(I) - 1. \end{aligned} \tag{13}$$

From (9), (12), and (13), we have

$$LB(I) \geq \frac{k}{k+1} OPT(I) - 1,$$

which implies $r_\infty(LB) \geq \frac{k}{k+1}$. ■

Theorem 2 reduces the task of deriving the asymptotic worst-case of a BP lower bound to that of finding: 1) the appropriate value of k such that the bound is able to solve to optimality the instances with volumes greater than $1/k$; 2) at least one instance where the asymptotic worst-case is tight. In the following, we apply this method to the L_3 bound.

Lemma 3 *Given an instance I of the BP problem with all the volumes $v_i > \frac{1}{3}$, the bound L_3 finds an optimal solution of the instance.*

Proof. Sort the items by non-increasing volumes v_i and consider the first item of the instance. It follows that $v_1 = \max_i \{v_i\}$. We have to check the following cases:

1. $|I| = 1$. The procedure $MTRP$ fails (e.g., it can not complete any bin), and $L_1 = L_2 = L_3 = 1$, that is, the optimal value.

2. $v_1 \in \left(\frac{3}{4}, 1\right]$. From the hypothesis, we know that $v_N > \frac{1}{3}$, so $v_1 + v_N > 1$. From the dominance criterion 1, the feasible set $F_1 = \{1\}$ belongs to an optimal solution and thus
$$\begin{aligned} &OPT(I) = 1 + OPT\left(I^1\right) \\ &I^1 = I \backslash F_1. \end{aligned} \tag{14}$$

3. $v_1 \in \left(\frac{1}{2}, \frac{3}{4}\right]$ and $|I| = 2$. If $v_2 \in \left(\frac{1}{2}, \frac{3}{4}\right]$, the two items must be loaded into 2 different bins and the $MTRP$ procedure builds them by means of the dominance criterion 1. Otherwise, the procedure $MTRP$ fails and $L_1 = L_2 = L_3 = 1$, that is, the optimal value.

4. $v_1 \in \left(\frac{1}{2}, \frac{3}{4}\right]$ and $|I| \geq 3$. By hypothesis, we know that $v_i > \frac{1}{3}, \forall i$. If $v_N \in \left(\frac{1}{2}, \frac{3}{4}\right]$, then the feasible set $F_1 = \{1\}$ belongs to an optimal solution by the dominance criterion 1, and (14) holds.
If $v_N \in \left(\frac{1}{3}, \frac{1}{2}\right]$, one of the following cases occurs. If the following inequalities hold
$$\begin{aligned} &1 - (v_1) \geq 0 \\ &1 - (v_1 + v_N) < 0, \end{aligned}$$
then, by the dominance criterion 1, the feasible set $F_1 = 1$ belongs to an optimal solution. Otherwise, the following inequalities are satisfied
$$\begin{aligned} &1 - (v_1 + v_N) \geq 0 \\ &1 - (v_1 + v_N + v_{N-1}) < 0 \end{aligned}$$
and, by the dominance criterion 2, it exists an item j such that the FS $F_1 = \{1, j\}$ is in an optimal solution, which implies
$$\begin{aligned} &OPT(I) = 1 + OPT\left(I^1\right) \\ &I^1 = I \backslash \{1, j\}. \end{aligned} \tag{15}$$

5. $v_1 \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $|I| = 2$. The procedure $MTRP$ fails and $L_1 = L_2 = L_3 = 1$, that is, the optimal value.

6. $v_1 \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $|I| \geq 3$. In this case, we have that v_N and $v_{N-1} \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and the following inequalities are satisfied

$$1 - (v_1 + v_N) \geq 0$$
$$1 - (v_1 + v_N + v_{N-1}) < 0.$$

By the dominance criterion 2, we know that it exists an item j such that the feasible set $F_1 = \{1, j\}$ belongs to an optimal solution, and (15) holds.

If we now consider the instance I^1, either the hypotheses of the theorem are satisfied or I^1 is empty. In the first case, cases 1-6 are applied recursively on I^1, building a new non-dominated feasible set F_2. Moreover, we can split I^1 in two sub-sets F_2 and I^2, where $I^2 = I^1 \backslash F_2$. The process can be continued, until an empty instance is found. Assuming the empty instance is found after K iterations, we have

$$OPT(I) = 1 + OPT\left(I^1\right) = 1 + 1 + OPT\left(I^2\right) = \sum_{k=1}^{K} 1 + OPT(I^{K+1}) =$$

$$\sum_{k=1}^{K} 1 + OPT(\{\emptyset\}) = K.$$

Then, the set I has been split into the feasible sets F_i and each F_i must be in an optimal solution. Then, the solution

$$F_1, ..., F_K$$

is an optimal solution of the instance I and $L_3(I) = OPT(I) = K$ is the optimal value of the instance I. ■

Lemma 4 *The asymptotic worst-case of L_3 is*

$$r_\infty(L_3) = \frac{3}{4}.$$

Proof. By definition, L_3 applies the $MTRP$ procedure to the instance I and, at each iteration, a new instance in obtained by discarding the item with the smallest volume. Thus, after a finite number of iterations, L_3 will consider an instance such that all the items are larger than $1/3$ and $L_3(I) \geq L_3(\{i \in I : v_i > 1/3\})$.

By Theorem 2 and Lemma 3, we have that

$$r_\infty(L_3) \geq \frac{3}{4}.$$

To prove that the worst-case ratio is tight, we consider the instances with $3k$ items of size $\frac{1}{4} + \delta$, $\delta > 0$ and 2 items of size $\frac{1}{2} - 2\delta$. These instances

need at least $k+1$ bins. For δ sufficiently small, we have that $MTRP$ is not able to find any FS. Indeed, the best pair is $\{1,2\}$, while the best triplet is $\{1,3,4\}$ and the dominance criterion 3 can not be applied. Thus, the bound is $L_3 = L_1 \leq \frac{3}{4}k + 1$. ■

Lemma 5 *Given an instance I of the BP problem with all the volumes $v_i > \frac{1}{2}$, the bound L_2 finds an optimal solution of the instance.*

Proof. If all the items have volumes $v_i > \frac{1}{2}$, then no item can be placed with any other item in the same bin. Thus, the optimal value of the objective function is $|I|$.

If we consider the bound L_2 with $\epsilon = 1/2$, $I_1(1/2) = I$ and $L_2 = |I_1| = |I|$. ■

Lemma 6 *The asymptotic worst-case of L_2 is*

$$r_\infty(L_2) = \frac{2}{3}.$$

Proof. Let us consider the sub-instance $I^* = \{i \in I : v_i > 1/2\}$. To prove that $L_2(I^*) \leq L_2(I)$, notice that, for each value of ϵ, the set $I(1-\epsilon, 1)$ contains the same items whether we are computing $L_2(I)$ or $L_2(I^*)$, while the set I_2 is such that $I_2(\epsilon, I^*) \subseteq I_2(\epsilon, I)$, where $I_2(\epsilon, I) = \{i \in I | \epsilon \leq v_i \leq 1-\epsilon\}$ and $I_2(\epsilon, I^*) = \{i \in I^* | \epsilon \leq v_i \leq 1-\epsilon\}$. Then, by Theorem 2 and Lemma 5,

$$r_\infty(L_2) \geq \frac{2}{3}.$$

To prove that the worst-case ratio is tight, we consider the instances with an even number of items n of volume $v_i = \frac{k+1}{3k}$, $k \geq 2$. The optimal solution is $n/2$, while, for $\epsilon = k+1$, $L_2 = \lceil n(k+1)/(3k) \rceil$. ■

Lemma 7 *Given an instance I of the BP problem with all the volumes $v_i > \frac{1}{3}$, the bound L_{2LLM} finds an optimal solution of the instance.*

Proof. Let us recall that I_3 is the set of items with volume $1/3 < v_i \leq 1/2$ which can not be assigned to a bin according to the adapted First Fit procedure. All volumes being greater than $1/3$, it is easy to see that $L_{2LLM} = \left\lceil \frac{|I_3|}{2} \right\rceil + \left| I(\frac{1}{2}, 1) \right|$.

With respect to the set I_3, one can build a feasible solution of the instance as follows:

- Assign each item in $I(\frac{1}{2}, 1)$ to a different bin, obtaining $\left| I(\frac{1}{2}, 1) \right|$ new bins;
- Place each item in $I(\frac{1}{3}, \frac{1}{2}) \backslash I_3$ in the same bin with the item in $I(\frac{1}{2}, 2)$ which has been assigned by the adapted First Fit procedure (no extra bin is added);
- Pair the items in I_3, obtaining $\lceil |I_3|/2 \rceil$ new bins.

Since a feasible solution with objective function equal to L_{2LLM} exists, the lower bound is able to find the optimal solution of the instance. ■

Lemma 8 *The asymptotic worst-case of* L_{2LLM} *is*

$$r_\infty(L_{2LLM}) = \frac{3}{4}.$$

Proof. Let us consider the sub-instance $I^\star = \{i \in I : v_i > 1/3\}$. The set I_3 contains the same items whether we compute $L_{2LLM}(I)$ or $L_{2LLM}(I^\star)$. Moreover, for each value of ϵ, the set $I(1-\epsilon, 1)$ contains the same items when we compute $L_{2LLM}(I)$ and $L_{2LLM}(I^\star)$, while the set I_2 is such that $I_2(\epsilon, I^\star) \subseteq I_2(\epsilon, I)$, where $I_2(\epsilon, I) = \{i \in I | \epsilon \leq v_i \leq 1-\epsilon\}$ and $I_2(\epsilon, I^\star) = \{i \in I^\star | \epsilon \leq v_i \leq 1-\epsilon\} = \{i \in I^\star | \epsilon \leq v_i \leq 1-\epsilon\, v_i > 1/3\}$. Thus, $L_{2LLM}(I^\star) \leq L_{2LLM}(I)$. Then, by Theorem 2 and Lemma 7,

$$r_\infty(L_{2LLM}) \geq \frac{3}{4}.$$

To prove that the worst-case ratio is tight, we consider the instances with n items of volume $v_i = \frac{k+1}{4k}$, $k > 3$. The optimal solution is $z = n/3$. Since $k > 3$, $v_i < 1/3$, and $L_{2LLM} = L(k+1) = \lceil n(k+1)/(4k) \rceil$, the ratio L_{2LLM}/z can be arbitrarily close to $3/4$ for k large. ■

Lemma 9 *The asymptotic worst-case of* $L_{FS}^{\star(p)}$ *is*

$$r_\infty\left(L_{FS}^{\star(p)}\right) = \frac{3}{4}.$$

Proof. By the definition of $L_{FS}^{\star(p)}$, the volume of each item is modified by dual feasible functions and L_1 is applied to the modified volumes. Thus, when $L_{FS}^{\star(2)}(I^\star)$ is computed for the sub-instance $I^\star = \{i \in I : v_i > 1/3\}$, the volumes of the items with $v_i \leq 1/3$ are not considered. Those items have, in $L_{FS}^{\star(2)}(I)$, a modified volume $v_i \geq 0$. Thus, $L_{FS}^{\star(2)}(I^\star) \leq L_{FS}^{\star(2)}(I)$.

Fekete and Schepers proved in [4] that $L_{FS}^{\star(2)}$ is able to solve to optimality all the instances where the minimum volume is greater than $\frac{1}{3}$. Then, by Theorem 2,

$$r_\infty\left(L_{FS}^{\star(2)}\right) \geq \frac{3}{4}.$$

To prove that the worst-case ratio is tight, we consider the instances with $n = 3k$ items of volume $v_i = \frac{1}{4} + \delta$, $\delta > 0$. In these instances, we use at least k bins. For δ sufficiently small, $L_{FS}^{\star(2)} = L_2 = L_1 \leq \frac{3}{4}k + 1$.

The same authors proved in [4] that for any value of p, the worst-case cannot be improved. ■

4 Conclusions

In this paper, we introduced a general theorem relative to the derivation of the asymptotic worst-case of BP lower bounds. We used this theorem to derive the worst-case ratio of the well-known lower bound L_3.

To further illustrate the possibilities offered by this methodology, we used it to obtain, rather directly, the existing asymptotic worst-case ratios of the bounds L_2, L_{2LLM}, and $L_{FS}^{\star(p)}$.

Additional illustrations of the results presented in this paper may be found in [3] and [9], where the asymptotic worst-case ratios of new lower bounds for the BP problem are derived.

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